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Meyers, C.; Roo, M. de

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NEW EXPLICIT INSTANTONS AND THE GEOMETRY OF THE PARAMETER SPACE

C. MEYERS and M. de ROO *
CERN, Geneva, Switzerland

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We obtain a geometrical description of the parameter space of instantons of topological charge k in an SU(n) gauge theory. We show how this space is related to a compact convex set of positive matrices. We give a characterization of points in the parameter space which correspond to embeddings. We derive an explicit formula for instantons of charge k in SU(4) which depend on $7k - 3$ parameters and are not reducible.

1. Introduction

In recent years the problem of constructing solutions of the Euclidean equations of motion of Yang-Mills theories has been attacked from different directions. On the one hand, people have attempted to solve the differential equations directly, using specific ansätze to simplify this formidable task. This approach has led to certain classes of explicit solutions [1], which all give gauge field configurations satisfying the self-duality property. On the other hand, the equations expressing self-duality have been analyzed by methods of algebraic geometry. By this second approach Atiyah, Hitchin, Drinfeld and Manin [2] have succeeded to obtain the general solution to the problem of constructing all self-dual field configurations on S^4 (instantons) for an arbitrary compact group. Their solution can be formulated in an extremely simple way which involves only elementary operations on matrices satisfying certain non-linear constraints. The dimensions of these matrices are related to the topological charge of the solution and to the order of the gauge group under consideration. The relation of the general construction to the formulation involving matrices has recently been made explicit by two groups [3,4].

Our aim in this paper is twofold. First of all we wish to obtain, for SU(n) gauge groups, the essential properties of the matrices which contain the instanton parameters. This will lead us to a formulation of the non-linear constraints which is different from that of refs. [3] and [4]. This allows us, for fixed charge k, to obtain a geometrical picture of the parameter space of SU(n) instantons for all n. In particular it turns out that this parameter space, before certain points leading to singular

* Present address: Instituut Lorentz, Leiden, the Netherlands.

solutions have been excluded, is a compact, convex set. These unwanted singular points belong to the boundary. The interior of this set consists of SU($2k$) instantons, the boundary being formed by solutions of SU(n) for $n < 2k$, embedded in SU($2k$). In our formulation we are able to characterize the reducibility of a gauge field configuration in terms of simple conditions on the space of parameters.

Secondly we want to present explicit instanton configurations of arbitrary topological charge for the gauge group SU(4). This calculation will illustrate the problems which occur if one constructs an explicit form for the gauge field A_μ from a point in the parameter space. These solutions have not been obtained previously by other methods.

We shall start by recalling the construction of instantons of Atiyah et al. [2] for SU(n), in the form of ref. [3]. The vector potential A_μ is given by

$$A_\mu = iV^+\partial_\mu V\,, \tag{1.1}$$

where V is a complex $(n + 2k) \times n$ matrix. For

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \tag{1.2}$$

to be self-dual, V must satisfy the following conditions:

$$V^+V = \mathbb{1}_n\,, \tag{1.3}$$

$$V^+\Delta = 0\,, \tag{1.4}$$

where $\Delta(x)$ is a $(n + 2k) \times 2k$ matrix, linear in x. $\Delta^+\Delta$ must be the tensor product of the unit matrix $\mathbb{1}_2$ and a $k \times k$ Hermitian matrix, and its inverse must exist. This symmetry condition can be expressed in the following way: let Δ_i, $i = 1, 2$ be the two $(n + 2k) \times k$ parts of Δ, then

$$\Delta_i = a_i + b_j x_{ji}\,, \tag{1.5}$$

where a_i and b_i are constant $(n + 2k) \times k$ complex matrices, and x is the quaternion

$$x = \begin{bmatrix} x_0 + ix_3 & i(x_1 - ix_2) \\ i(x_1 + ix_2) & x_0 - ix_3 \end{bmatrix}, \tag{1.6}$$

then

$$a_i^+a_j = \mu\delta_{ij}\,, \qquad b_i^+b_j = \nu\delta_{ij}\,, \qquad \epsilon_{ij}b_j^+a_k = \epsilon_{jk}a_j^+b_i\,, \tag{1.7}$$

with $\epsilon_{ij} = -\epsilon_{ji}$, $\epsilon_{12} = 1$. Anti-self-dual field configurations are obtained by replacing x by x^+ everywhere.

It has been proved by Atiyah et al. [2] that this construction gives all self-dual solutions for SU(n) of charge k. The same authors also show that the construction is essentially unique: to one A_μ corresponds only one pair $\{V, \Delta\}$ up to equivalences which we shall now discuss.

In (1.3) and (1.4) V can of course be multiplied on the right by any element of SU(n) (x-dependent) without changing its defining properties. This corresponds to a gauge transformation of A_μ. We can also change Δ and V such that A_μ remains the same: if we transform

$$a_i \to Ua_iK\,, \qquad b_i \to Ub_iK\,, \qquad V \to UV\,, \tag{1.8}$$

where U is a unitary $(n+2k)\times(n+2k)$ matrix, and K an arbitrary real invertible $k\times k$ matrix, all regularity and symmetry conditions are still satisfied, and the transformation matrices do not play any role in the calculation of A_μ. This means that there is a certain arbitrariness in the parameters which appear in the vectors a_i and b_i, not all of them are relevant. We shall come back to this in sect. 2, where we present a somewhat different way to express the constraints on Δ. There we shall obtain a geometrical description of the parameter space for fixed topological charge k. In sect. 3 we consider the action of the conformal group on this space. In sect. 4 we discuss the problem of finding a solution of a smaller group then expected, and we characterize these embeddings by properties of the parameter space. In sect. 5 we shall discuss the problems one encounters in finding points belonging to the parameter space, and in constructing A_μ explicitly from such points. We present solutions of SU(4) for arbitrary k, which depend on $7k-3$ parameters.

2. The parameter space

As we have seen in sect. 1, the construction of instantons involves a matrix Δ, satisfying certain conditions, and containing all the relevant parameters of the solution. We shall present a different way of imposing the conditions (1.7). Although the result is the same, our method gives a clearer insight in the structure of the parameter space.

Let us group the vectors a_i, b_i in a $(2k+n)\times 4k$ matrix C and construct the matrix M:

$$M = C^+C = \begin{bmatrix} a_1^+a_1 & a_1^+a_2 & a_1^+b_1 & a_1^+b_2 \\ a_2^+a_1 & a_2^+a_2 & a_2^+b_1 & a_2^+b_2 \\ b_1^+a_1 & b_1^+a_2 & b_1^+b_1 & b_1^+b_2 \\ b_2^+a_1 & b_2^+a_2 & b_2^+b_1 & b_2^+b_2 \end{bmatrix} = \begin{bmatrix} \mu & 0 & \sigma & \rho \\ 0 & \mu & -\rho^+ & \sigma^+ \\ \sigma^+ & -\rho & \nu & 0 \\ \rho^+ & \sigma & 0 & \nu \end{bmatrix}. \tag{2.1}$$

M is therefore a positive Hermitian $4k\times 4k$ matrix of rank $n+2k$, with the blockwise symmetry properties shown in (2.1).

Since Atiyah et al. [2] have shown that their construction gives the complete set of self-dual Yang-Mills fields, we can associate a matrix M to any solution A_μ. We shall show that from the set of matrices M with the properties mentioned above we can recover the complete set of solutions A_μ. To do this we reconstruct Δ from

M. Since M is positive and of rank $n + 2k$, we can always find a matrix C, such that C has dimension $(n + 2k) \times 4k$, and $C^+C = M$. C is unique up to a unitary transformation $C \to UC$, where $U \in U(n + 2k)$. We take Δ to be

$$\Delta = C\chi\,, \qquad \chi = \begin{bmatrix} \mathbb{1}_{2k} \\ X \end{bmatrix}, \tag{2.2}$$

where X is the tensor-product of $\mathbb{1}_k$ with the quaternion (1.6). The arbitrariness of C, and therefore of Δ, does not change A_μ, (see also (1.8)) so that indeed we can associate a unique A_μ (of course up to a gauge transformation) to any matrix M for which the corresponding $\Delta^+\Delta$ is invertible for all x. However, different matrices M may give the same A_μ. This is of course because of the transformations K in (1.8) which we have not yet taken into account. The transformation K on the vectors a_i, b_i amounts to the following transformation of the matrix C:

$$C \to C\Gamma\,, \qquad \Gamma = \begin{bmatrix} K & & & \\ & K & & \\ & & K & \\ & & & K \end{bmatrix},$$

and therefore M is conjugated by the regular matrix Γ of the form shown above:

$$M \to M' = \Gamma^+ M \Gamma\,. \tag{2.3}$$

Thus all matrices M which are related by (2.3) will lead to the same A_μ, and must therefore be considered equivalent.

We are now in a position to count the number of parameters. For a Hermitian matrix of dimension N to be of rank r one must impose $(N - r)^2$ conditions. The transformations Γ eliminate $2k^2 - 1$ parameters (since there is one phase which is not effective). The number of parameters is therefore $6k^2 - (2k - n)^2 - 2k^2 + 1 = 4nk - n^2 + 1$ if $n \leqslant 2k$. For $n \geqslant 2k$ there are no rank constraints on the matrix M and the number of parameters is $4k^2 + 1$.

In sect. 1 we mentioned that $\Delta^+\Delta = \chi^+ M\chi$ must be invertible for all x. The reason is clear from (1.4): if Δ has rank smaller than $2k$ for some x, V is no longer determined up to a unitary transformation, and this will give rise to a singularity of A_μ at finite x. We must therefore exclude matrices M which have μ singular, since they give rise to a singularity at $x = 0$. These are not the only matrices for which $\chi^+ M\chi$ can be singular for some x, unfortunately the set of such matrices is not explicitly known. Another problem arises when the matrix ν is singular. This does not necessarily give a singularity of $\chi^+ M\chi$ at finite x, but can lead to solutions in which A_μ does not have the proper topological charge [4]. As an example consider the solution for $k = 1$ of Belavin et al. [1]:

$$A_\mu = \frac{x^2}{x^2 + \lambda^2}\, iU\partial_\mu U^{-1}\,, \tag{2.4}$$

where U is the usual $k = 1$ gauge transformation, singular at $x = 0$. The limit $\lambda \to 0$ corresponds to a singularity of μ in our matrix M, and A_μ develops a singularity at $x = 0$. The limit $\lambda \to \infty$ is the limit in which ν becomes singular, and $A_\mu \to 0$ everywhere. We must exclude all M for which $\chi^+ M\chi$ or ν are singular matrices.

The parameter space for instantons of charge k in SU(n) is therefore the space of $4k \times 4k$ matrices M, Hermitian and positive, of rank $n + 2k$, with the symmetry (2.1), with all M related by Γ (as in (2.3)) identified, and from which all points corresponding to the singularities mentioned above are excluded.

We shall now give a geometrical description of this parameter space. Given that μ and ν are strictly positive we can use the transformations Γ to put $\mu + \nu = \mathbb{1}/2k$. The points which remain to be identified are those which are related by matrices Γ with $K = U \in$ SU(k).

Consider now the set $\mathcal{D}$ of matrices M, tr $M = 1, M \geqslant 0$, with the symmetry (2.1), and with $\mu + \nu = \mathbb{1}/2k$. This is clearly a compact convex set, which is a subset of a $5k^2$ dimensional vector space. All instantons of charge k for SU(n) with n arbitrary are contained in this compact set, of which we must now eliminate the points leading to singular solutions. The boundary $\partial\mathcal{D}$ consists of those M for which the rank is smaller than $4k$. Any M which has either ν or $\chi^+ M\chi$ singular must have rank less than $4k$, and therefore all points we have to exclude are on the boundary. The remaining set is not compact. The parameter space is the orbit space associated with the action of the group SU(k) on this remaining set.

The interior of $\mathcal{D}$ consists of all M which are strictly positive, and correspond to instanton configurations of SU($2k$). All instanton of charge k for SU($n > 2k$) are in fact canonical * embeddings of SU($2k$) in SU(n). On the boundary of $\mathcal{D}$ we find the canonical embeddings of SU(n), $n < 2k$, in SU($2k$). Any other embeddings of these and other groups, which are in the interior and on the boundary of $\mathcal{D}$ will be discussed in sect. 4.

The convexity of the set $\mathcal{D}$ allows us to construct from any two instanton solutions an infinity of solutions of SU($2k$). In practice, one is interested in finding solutions for any k for a given SU(n) group, which means that one looks for points on the boundary of $\mathcal{D}$, for which the convexity property is of no help.

3. The action of the conformal group on the parameter space

The problem of finding all the instanton solutions for a group SU(n) has been reformulated as a problem involving matrices of certain rank and symmetry. It is therefore important to find the group of transformations which transforms any matrix of this type into another one. Since it is known that the conformal group O(5,1) in Euclidean space transforms one solution of the Yang-Mills equations into

* A canonical embedding of SU(n) in SU(p), $p > n$, is, in matrix form, in the upper left-hand $n \times n$ corner.

another, we shall find the action of this group on the manifold of matrices M.

To find the action of co-ordinate transformations on M it is useful to formulate the construction of A_μ in a different, but equivalent way. Let us introduce a matrix W, of dimension $4k \times n$, which satisfies:

$$W^+MW = \mathbb{1}_n\,, \qquad W^+M\chi = 0\,. \tag{3.1}$$

If M has all the properties mentioned in sect. 2, with χ as defined in (2.2), then

$$A_\mu = iW^+M\partial_\mu W \tag{3.2}$$

is a self-dual field configuration. It is easy to see that this construction is equivalent to that given in the Introduction. To any pair $\{W, M\}$ we can trivially associate a pair $\{V, \Delta\}$ by direct calculation. On the other hand, for a given $\{V, \Delta\}$ we know the matrix C which was used in (2.1) to calculate M, therefore we know M, and for W we can take any solution of $V = CW$ (there is at least one solution because the rank of C is $n + 2k$).

It is well-known that if one represents space-time points by quaternions the action of the conformal group corresponds to homographic transformations in quaternion space:

$$x \to x' = (\gamma + \delta x)(\alpha + \beta x)^{-1}\,, \tag{3.3}$$

where α, β, γ and δ are quaternions. To define the action of the conformal group on the matrices χ of eq. (2.2) we introduce the $4k \times 4k$ matrices

$$T = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \otimes \mathbb{1}_k\,. \tag{3.4}$$

If we multiply χ on the left by T we find

$$T\chi = \begin{bmatrix} \mathbb{1}_{2k} \\ X' \end{bmatrix} \{(\alpha + \beta x) \otimes \mathbb{1}_k\} = \chi'\{(\alpha + \beta x) \otimes \mathbb{1}_k\}\,. \tag{3.5}$$

The transformation of eq. (3.1) is then

$$W^+T^+[T^{-1+}MT^{-1}]\,\chi'\{(\alpha + \beta x) \otimes \mathbb{1}_k\} = 0\,, \tag{3.6}$$

so that M transforms as

$$M \to M' = T^+MT\,, \tag{3.7}$$

where T is any regular matrix of the type (3.4). So the rank of M does not change, and it is easy to check that this is also true for the symmetry properties (2.1).

A representation of the conformal group is formed by the matrices T of (3.4), if we identify all T which are related to each other by real scalar multiplication. This obviously is a 15 parameter group. It is interesting to consider the action of the subgroups of O(5,1) on M, since this may help to give a physical interpretation of the parameters appearing in M. The compact part is $O(5) \simeq Sp(4)$, which contains the Euclidean Lorentz group O(4) of which the elements are block diagonal matrices T. Translations are generated by matrices T with $\alpha = \delta = \mathbb{1}_2$, $\beta = 0$, dilatations by $\gamma = \beta = 0$, $\alpha = \mathbb{1}_2$, $\delta = \lambda \mathbb{1}_2$.

We put M in the form

$$M = \begin{bmatrix} N & R^+ \\ R & \mathbb{1}_{2k} \end{bmatrix} \tag{3.8}$$

by a transformation (2.3), which is possible for all M since ν is regular. One can easily verify that the action of the Lorentz group, the translations and the dilatations on R is the same as their action on X, indicating that R contains the generalized positions of multi-instantons.

The action (3.7) on M constructs linear combinations of μ, ν, ρ and σ which have the same symmetry properties. In this way it is possible to interchange μ and ν, so that the problems of singularities of μ and ν, which we discussed as separate cases in sect. 2, are conformally related. It is also clear why, for $k = 1$, the conformal transformations do not give anything new: since μ, ν, ρ and σ are numbers, any linear combination with the same symmetry properties amounts to a redefinition of the original parameters.

Presumably the conformal group is the largest group of transformation of the parameter space. Certain subsets of the parameter space have a larger transformation group, of which the number of parameters depend on k. This implies that starting from a point of the parameters space which has some particular properties we can use a group of transformations, containing the conformal group, to generate a new set of points.

4. Embeddings

In this section we shall discuss some group theoretical properties of the solutions that are obtained with the construction of Atiyah et al. We shall take fixed k, and consider $n \leqslant 2k$ (since $n > 2k$ leads to embeddings of SU($2k$) in SU(n)). We have already seen in sect. 2 that on the boundary of the domain of matrices M with rank $n + 2k$ we find embeddings of SU(n) in SU($2k$). However, in the interior of the domain (rank $M = 4k$) not all solutions will give rise to an irreducible gauge field of SU($2k$).

Before considering properties of the general case let us look in detail at $k = 2$ [SU(4)] to illustrate the kind of situation which can occur. For $k = 2$ the boundary of the domain of matrices M contains solutions of SU(2) and SU(3). These embeddings are characterized by the fact that their Dynkin index is 1. In general, if a gauge field configuration of charge q for a group G is embedded in SU(n), the charge as calculated in SU(n) will be $k = jq$, where j, the Dynkin index, is an integer which distinguishes in most cases the inequivalent embeddings of G in SU(n). For details on Dynkin indices and the complete characterization of embeddings we refer to ref. [5] and references therein. We consider $k = 2$, and therefore the only embeddings which can possibly be obtained are those for which $j = 1$ or 2. This means, for instance, that the $k = 2$ solutions of SU(3), which we obtain on the boundary, do not contain any solution of O(3) embedded in SU(3), since the O(3) embeddings has $j = 4$. There are other subgroups of SU(4) which can be excluded by this argument, as for instance, the SU(2) of index 10. In fact, for $k = 2$ [SU(4)] the only possible remaining embedding is that of Sp(4) $\simeq$ O(5), which has index 1, and some of its subgroups. Sp($2n$) is the group of $n \times n$ unitary quaternionic matrices.

If one is interested in finding explicit instanton solutions of the group SU(n), it is therefore important to find the conditions on the matrix M which distinguish the SU(n) solutions from the embeddings. So we want to characterize embeddings by a set of supplementary conditions on M. Since the rank characterizes the unitary group we are considering, the supplementary conditions must involve the symmetry properties of M. There are two ways to find these conditions: they can be obtained from the condition that A_μ is reducible, or we can use the properties of the instanton construction for the embedded groups. The second method is clearly the easiest, as we can obtain the conditions by a direct calculation. Also it is known that the reducibility of A_μ is difficult to characterize by A_μ itself [6].

The problem of characterizing embeddings of all subgroups of SU($2k$) is too complicated to be treated completely. Therefore we shall consider only certain subgroups which are relevant in constructing solutions. We start by considering canonical embeddings (i.e. of index 1) of Sp($2p$) in SU($2k$) for $p \leqslant k$, and we shall then discuss groups $G_1 \times G_2$, with G_1 and G_2 both simple and of index 1, and of type Sp or SU. Finally we give some properties of O(n) instantons.

For the Sp($2p$) solutions we shall not repeat the construction of the solutions, from ref. [3] it is easy to obtain the appropriate vectors a_i and b_i and find the following result: if the matrix M of (2.1) has rank $2p + 2k$ and satisfies the supplementary condition that μ, ν, ρ and σ are symmetric matrices, then the resulting gauge field A_μ will be an embedding of an Sp($2p$) gauge field in SU($2p$).

This is only a sufficient condition, since the conjugations (2.3) which do not change A_μ, can change the symmetry properties of μ, ν, ρ and σ. A complete statement is:

The matrix M of (2.1), rank ($2p + 2k$) corresponds to a gauge field of Sp($2p$) embedded in SU($2p$) if and only if there exists a transformation Γ (2.3) which symmetrizes μ, ν, ρ and σ. The "only if" part of this statement refers to the fact that the self-dual gauge field configuration determines Δ, and therefore M, in an essentially unique way [2].

It is easy to generalize this for embeddings of groups which are direct products of simple groups, of index 1 in SU($2k$). Let $G_1 \times G_2$ be such a subgroup of SU($2k$), then the matrices μ, ν, ρ and σ are simultaneously block diagonal (up to transformations Γ):

$$\mu = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix},$$

and similarly for ρ, σ, ν, and such that the blocks for each of these matrices have the same dimensions, k_1 and k_2, respectively. M is then the direct product of two matrices M_1 and M_2 which have rank $2k_1 + n_1$ and $2k_2 + n_2$, respectively, where $2k + n_1 + n_2$ is the rank of M. We see that the problem of recognizing embeddings in the parameter space is analogous to but simpler than the problem of recognizing embeddings on A_μ itself: the task is to recognize certain symmetry properties which may have been distorted by a conjugation.

In particular, for our example of $k = 2$, SU(4) we see that μ, ν, ρ and σ symmetric (up to a transformation Γ) leads to an Sp(4) solution. Such a solution will in general be in the interior of the domain of matrices M. The solution can reduce further to SU(2) $\times$ SU(2), when not only the conditions for Sp(4) are satisfied, but also μ, ν, ρ and σ are blockdiagonal. Finally, if they are block diagonal with identical blocks, the solution reduces to SU(2) of index 2, and the solution must be the one of Belavin et al. [1] of charge 1 embedded with index 2 in SU(4).

Embeddings of groups O(n) in SU(n) are considerably more difficult to characterize. We shall consider only the groups O(n) in the representation of $n \times n$ orthogonal matrices, so that we find O(n) as an embedding in SU(n). For the construction which starts from k, $n \leqslant 2k$ we shall not often encounter these O(n) groups. In fact, they all have index 2, except for $n = 3$, when the index is 4. So if k is odd, we are sure that the SU($2k$) solutions are not embeddings of O($2k$) in the standard representation. If we consider the domain of matrices M with rank $4k$, i.e., the interior of the domain $\mathcal{D}$, we will never encounter configurations which are irreducible solutions of O(n), $n < 2k$ in the standard representation. These solutions are always embedded in SU(n) with $n < 2k$, and therefore must be on the boundary of the domain. The smallest topological charge for which this occurs is $k = 4$, when we have the $q = 1$ instanton in O(3) embedded in SU(3), on the boundary of the set of matrices M for SU(8).

5. The construction of explicit solutions

To find an explicit self-dual gauge field configuration we must start by finding a matrix M which satisfies the constraints discussed in sect. 2. We have not been able to find a parametrization of M which satisfies all positivity, rank and symmetry conditions and contains all parameters, except for $k = 1$. We must therefore choose μ, ν, ρ and σ such that some of the constraints simplify, and in this process we lose some parameters. We shall find a natural way to simplify these matrices, which will lead to solutions which are generalizations of 't Hooft's [1] SU(2) instantons to SU(n), for n even. We then calculate such solutions for SU(4).

First of all we shall use the transformations Γ of (2.3) to choose $\nu = \mathbb{1}_k$. This means that the general form of M becomes

$$M = \begin{bmatrix} N & R \\ R & \mathbb{1}_{2k} \end{bmatrix}, \qquad R^+ = \begin{bmatrix} \sigma & \rho \\ -\rho^+ & \sigma^+ \end{bmatrix}, \qquad N = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}. \tag{5.1}$$

M has dimension $4k \times 4k$ and is positive of rank $n + 2k$ for SU(n). Such a matrix M has rank $n + 2k$ and is positive if and only if

$$N - R^+R = B^+B \tag{5.2}$$

is a positive $2k \times 2k$ matrix of rank n. We can therefore choose for B a matrix of dimension $n \times 2k$. If M and R are such that a matrix B of the correct rank can be obtained, we can easily find the corresponding matrix $\Delta(x)$:

$$\Delta(x) = \begin{bmatrix} B & 0 \\ R & \mathbb{1}_{2k} \end{bmatrix} \chi = \begin{bmatrix} B \\ R + X \end{bmatrix} . \tag{5.3}$$

Let us first analyze (5.2) in more detail. For ease of notation we introduce $k \times k$ Hermitian matrices a_μ, which we shall call generalized position matrices, to write R as a tensor product:

$$R = -i\sigma_\mu^+ \otimes a_\mu , \tag{5.4}$$

where $\sigma_0 = i\mathbb{1}_2$, and σ_a are the Pauli matrices. It is easy to check that this tensor product has exactly the symmetry properties (5.1). With the same notation, we can write

$$X = i\sigma_\mu^+ \otimes x_\mu \mathbb{1}_k . \tag{5.5}$$

We find that

$$R^+R = \mathbb{1}_2 \otimes a_\mu \; a_\mu + \tfrac{1}{2} i\bar{\eta}_{a\mu\nu} \sigma_a \otimes [a_\mu, a_\nu] . \tag{5.6}$$

If we choose the position matrices a_μ such that they commute among each other the equation (5.2) simplifies considerably. Of course this choice is equivalent to taking the a_μ all diagonal. Equation (5.2) becomes:

$$\mathbb{1}_2 \otimes (\mu - a_\mu \; a_\mu) = B^+B . \tag{5.7}$$

Note that the rank of the left-hand side is always even, so that this simplification is possible only for SU(n) with n even. We now parametrize B by writing

$$B = \mathbb{1}_2 \otimes \Lambda \tag{5.8}$$

where Λ is an arbitrary $p \times k$ complex matrix, with $n = 2p$. Thus (5.7) determines μ, and M constructed from R and μ as in (5.1), is now certainly positive and of rank $n + 2k$. As we shall see in the following this is a generalization of 't Hooft's solution to SU($2p$). The matrices a_μ, being diagonal, contain precisely the positions of the k instantons. The matrix Λ contains the widths, and some relative orientations.

Let us count the number of parameters in this solution. First we have $4k$ position parameters. Since $\mu - a_\mu \; a_\mu$ must be of rank p ($n = 2p$), μ contains only $2pk - p^2$ parameters. We shall assume that the positions are independent. The only conjugations of SU(k) which conserve the particular form we have chosen are those by the

U(1) × U(1) ... × U(1) ($k - 1$ times) subgroup, which can eliminate $k - 1$ parameters. Therefore the total number of parameters in our restricted class of SU($n = 2p$) solutions is $(3 + 2p)\,k - p^2 + 1$. By conformal transformation (see sect. 3) we can add at most 4 parameters, since only the special conformal transformations give non-diagonal $\mathfrak{a}_\mu$.

Because of the block structure of eq. (5.7) the rank n obtained with the choice of diagonal position matrices is always even. This implies that the generalization of 't Hooft's solution to SU(3) is likely to be much more complicated than the rather simple result we shall now give for SU(4). It is in fact possible to obtain SU(3) solutions by choosing the position matrices such that some of the commutators in (5.6) are different from zero. Since the result of this calculation cannot be written down in simple form, we shall not present it here.

We shall solve the equations (1.3) and (1.4) for SU(4) with k arbitrary and with Δ as in eq. (5.3), and B given by (5.8). The matrix Λ has the form

$$\Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1k} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2k} \end{bmatrix} , \tag{5.9}$$

where the λ_{ij} are complex numbers. The formula (1.1) for A_μ can be written as

$$A_\mu = i \sum_{j=1}^{k+2} V_j^\dagger \partial_\mu V_j , \tag{5.10}$$

where the V_j are 2×4 matrices *. Since we can form these matrices by taking rows of V together in any order we choose to do this in such a way that the corresponding rearrangement of Δ forms quaternionic 2×2 position matrices $x - a_j$. The equations (1.3) and (1.4) can then be easily solved. To write the final result, let us first introduce

$$\rho_{ij} = \delta_{ij} + \sum_{l=1}^{k} \frac{\lambda_{il}\lambda_{jl}^*}{(x - a_l)^2} , \tag{5.11}$$

which are generalizations of the usual function ρ appearing in multi-instanton solutions of SU(2). We then define SU(4) group elements

$$W_j = \begin{bmatrix} \alpha_j^* \mathbb{1}_2 & \beta_j^* \mathbb{1}_2 \\ -\beta_j \mathbb{1}_2 & \alpha_j \mathbb{1}_2 \end{bmatrix} \frac{1}{\sqrt{|\alpha_j|^2 + |\beta_j|^2}} , \qquad j = 1, ..., k , \tag{5.12}$$

with

$$\alpha_j = \frac{\lambda_{1j}}{\rho_{11}} - \frac{\rho_{12}}{\rho_{11}\rho_{22}} \lambda_{2j} , \qquad \beta_j = \frac{\sqrt{\rho_{11}\rho_{22} - |\rho_{12}|^2}}{\rho_{11}\rho_{22}} \lambda_{2j} , \tag{5.13}$$

* For SU(2) the V_j are 2×2 matrices, which means that any SU(2) solution is a weighted sum of $k + 1$ pure gauge terms.

which can be interpreted as the orientation of the vectors V_j appearing in (5.10). The resulting formula for A_μ is then

$$A_\mu = \sum_{j=1}^{k} \rho_j W_j^+ \begin{pmatrix} iU_j^+ \partial_\mu U_j & 0 \\ 0 & 0 \end{pmatrix} W_j + \frac{i}{2} \frac{\rho_{22}}{\sqrt{\rho_{11}\rho_{22} - |\rho_{12}|^2}} \partial_\mu \begin{bmatrix} 0 & \frac{\rho_{12}}{\rho_{22}} \mathbb{1}_2 \\ -\frac{\rho_{12}^*}{\rho_{22}} \mathbb{1}_2 & 0 \end{bmatrix} . \tag{5.14}$$

The U_j's are the well-known SU(2) elements

$$U_j = \frac{i\sigma_\mu^+ (x^\mu - a_j^\mu)}{\sqrt{(x - a_j)^2}} , \tag{5.15}$$

and the weight factors ρ_j are

$$\rho_j = \frac{\rho_{11}\rho_{22}}{\rho_{11}\rho_{22} - |\rho_{12}|^2} \left[\frac{1}{\rho_{11}} \frac{|\lambda_{1j}|^2}{(x - a_j)^2} + \frac{1}{\rho_{22}} \frac{|\lambda_{2j}|^2}{(x - a_j)^2} - 2 \frac{\mathrm{Re}(\lambda_{1j}\lambda_{2j}^*\rho_{12}^*)}{\rho_{11}\rho_{22}(x - a_j)^2} \right] . \tag{5.16}$$

We see in (5.14) that A_μ is essentially the weighted sum of k terms, each of which is an SU(2) instanton of charge 1, oriented in SU(4) by the group elements W_j. Unfortunately, the second term, which we have not been able to eliminate by gauge transformations and redefinitions, spoils this simple interpretation.

It can be easily checked that the limit $\lambda_{2j} \to 0$ for all j gives the 't Hooft SU(2) instanton solution [1] embedded in SU(4). The same thing happens if the vectors λ_1 and λ_2 are parallel.

We still have to show that in all other cases the field configuration A_μ is not in an embedding of some other group in SU(4). The only serious candidate for such a group is Sp(4) and we must therefore check that the matrices μ, ρ, and σ in (5.1), corresponding to our choice of parameters, cannot be symmetrized by an SU(k) transformation (see sect. 4).

With our choice ρ and σ are already diagonal. The only SU(k) elements which by conjugation transform an arbitrary diagonal matrix into a symmetric one are of the form

$$U = DO ,$$

where D is a unitary diagonal operator and O an element of O(k). Then it is easy to derive that a necessary and sufficient condition for an Sp(4) embedding is that $D^+\mu D$ is a symmetric matrix. We now use the fact that μ has the form

$$\mu = a_\mu \, a_\mu + \Lambda^+\Lambda . \tag{5.17}$$

Since a_μ a_μ is diagonal, the condition that $D^+\mu D$ is symmetric is equivalent to the condition

$$D^+\Lambda^+\Lambda D = S\,, \tag{5.18}$$

where S is symmetric. Λ has been constructed with two arbitrary complex vectors. Condition (5.18) implies that there must exist a diagonal unitary transformation D which transforms the two complex eigenvectors of $\Lambda^+\Lambda$ in real vectors. It is always possible to choose D such that one eigenvector is real. If the second vector is then also real, (5.18) is satisfied and we have an embedding. Since we have chosen two completely arbitrary complex vectors, this will in general not be the case. To have explicit solutions of O(5) $\simeq$ Sp(4) it is therefore sufficient to choose the matrix (5.9) to be real, and to use the same formula (5.14) for A_μ. This O(5) solution depends on $6k - 1$ parameters.

In principle it is possible to generalize (5.14) to groups SU($2p$) with p arbitrary. The difficulty lies in the normalization condition (1.2), which involves the explicit orthonormalization of p complex vectors. For large p this has to be sone iteratively, which makes it impossible to write down A_μ in compact form.

Added note

After completion of this work we have received a paper by Drinfeld and Manin, [7], in which a similar presentation of the instanton parameter space can be found.

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